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# Generalized eigenvalue passivity assessment of descriptor systems with applications to symmetric and singular systems

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## SUMMARY

This paper proposes general eigenvalue-based passivity tests for descriptor state space systems, extending the already available tests for standard state space systems. Generalized eigenvalue tests are proposed, to identify passivity violations in the positive-real and bounded-real cases. The important practical case of singular descriptor state space systems is also treated. The proposed passivity tests for descriptor systems can be used in the important case of symmetric (reciprocal) systems, since symmetric systems can always be cast in an explicitly reciprocal descriptor state space format. Also, passivity violation assessment methods are developed in the important case of strictly proper symmetric systems, since in that case the classical test matrices for passivity assessment are undefined. Finally, some pertinent numerical examples are given to demonstrate the usefulness and scope of the different passivity violation assessment tests. Copyright © 2011 John Wiley & Sons, Ltd.

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**KEY WORDS:** Descriptor systems; symmetric systems; positive-real lemma; bounded-real lemma; passivity violations

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## 1. INTRODUCTION

The characterization of linear time-invariant (LTI) systems from measured or simulated frequency domain data is an important issue in microwave modelling techniques. The LTI input can be in the form of immittance — (Y) or (Z) parameter data —, but also often scattering parameter (S) data are given. The LTI output is in state space realization format  $(A, B, C, D)$  or descriptor state space realization format  $(A, B, C, D, E)$ , since the physics of a system is often better described by introducing an additional descriptor matrix  $E$  [1]. Descriptor systems have been widely used in different modelling fields such as robotics [2] and micro-electromechanical systems [3]. Most importantly, many circuit modelling techniques, such as modified nodal analysis (MNA) [4], naturally produce models in a descriptor state space format. Although the LTI state space model under scrutiny must mandatorily be stable (or even Hurwitz stable), this alone proves unsatisfactory in simulations because the passivity of the model is not assured. Several methods [5] have been proposed in the literature aiming at enforcing passivity by a perturbation of the state space model parameters. All these methods require the ability to assess the passivity violations of the model. For that purpose, it is common practice to calculate the eigenvalues of a so-called Hamiltonian matrix or test matrix associated with the LTI state-space model [6]. The purely imaginary eigenvalues of the test matrix define frequency boundaries for passivity violations, thereby allowing to pinpoint frequency intervals where the model is non-passive. While the test matrix-based passivity assessment for the standard state space realization format  $(A, B, C, D)$  has been intensively treated, its counterpart for the descriptor state space realization format  $(A, B, C, D, E)$  is not that well-developed, although interesting methods were presented in [7, 8, 9, 10]. Here we propose different passivity assessment methods, extending the already available tests for standard state space systems. Singular descriptor systems in the positive-real ( $D + D^T$  singular) and bounded-real ( $D^T D - I$  singular) cases are discussed. We propose generalized eigenvalue techniques based on ideas in [11] and frequency inversion-based approaches for singular descriptor systems, which are different from the equivalent model conversion-based approaches used in [7, 8], where a shifting coefficient  $\alpha$

must be chosen properly. No additional choice of any parameter is needed in our approaches. The paper [9] describes only the positive-real case (admittance and impedance parameters) and does not explicitly and clearly discuss singular descriptor systems in the positive-real ( $D + D^T$  singular) and bounded-real ( $D^T D - I$  singular) cases. In addition, we also accurately describe the case of strictly proper symmetric systems, which are often encountered in practical applications.

This paper is organized as follows. In section 2 we derive the generalized eigenvalue equations pertaining to passivity violations for descriptor state space systems in the immittance (Y, Z) and scattering (S) cases. In section 3 we propose passivity tests based on generalized eigenvalues [11] and frequency inversion [12], to deal with the important practical case of singular descriptor state space systems in the positive-real (PRV) and bounded-real violations (BRV) cases. In section 4 we apply the proposed passivity tests for descriptor systems to the special case of symmetric (reciprocal) systems, since these systems can always be cast in an explicitly reciprocal descriptor state space format [13]. Also, passivity violation assessment methods are developed in the important case of strictly proper symmetric systems, since the classical Hamiltonian matrices are undefined in that case. Finally, in section 5, five pertinent real-world numerical examples are given to demonstrate the usefulness and scope of the different passivity violation assessment methods.

## 2. DESCRIPTOR SYSTEMS

Consider the descriptor state space system with realization

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where  $B \neq 0$  and  $C \neq 0$  are respectively  $n \times p$  and  $p \times n$  real matrices and  $A, E \neq 0$  are  $n \times n$  real matrices such that  $sE - A$  is a regular matrix pencil, i.e.,  $\det(sE - A) = 0$  has a finite number of  $s$  values as solutions. For the system to be positive-real, it is required that the  $p \times p$  transfer function

$$H(s) = C(sE - A)^{-1}B + D$$

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is analytic in the open right half-plane  $\Re[s] > 0$ , such that

$$G(i\omega) = \frac{1}{2} (H(i\omega) + H(-i\omega)^T) \geq 0 \quad \forall \omega \in \mathbb{R}$$

Since  $G(i\omega)$  is Hermitian, its eigenvalues are real, and the condition for positive-realness can be written as

$$\inf_{\omega \in \mathbb{R}} \lambda_{\min}[G(i\omega)] \geq 0$$

Here  $\lambda_{\min}[G(i\omega)]$  is the minimum eigenvalue of  $G(i\omega)$ .

For a system to be bounded-real on the other hand, it is required that  $H(s)$  is analytic in the open right half-plane  $\Re[s] > 0$ , such that

$$\|H\|_{\infty} = \sup_{\omega \in \mathbb{R}} \|H(i\omega)\|_2 = \sup_{\omega \in \mathbb{R}} \sigma_{\max}[H(i\omega)] \leq 1$$

Here  $\sigma_{\max}[H(i\omega)]$  is the maximum singular value of  $H(i\omega)$ . We notice that in the case of an invertible  $E$  matrix, the descriptor system can be converted into a standard  $A, B, C, D$  state space form

$$\dot{x} = E^{-1}Ax + E^{-1}Bu$$

$$y = Cx + Du$$

but such inversion is computationally expensive for high order systems. When  $E$  is singular, the conversion of the descriptor system into a standard state space form can be performed by using the SVD coordinates-based approach in [14] or computing a Weierstrass-like form of the pencil matrix [15]. In this paper we propose a number of straightforward passivity violation tests that do not require a conversion of the descriptor state space to standard state space form.

### 2.1. Positive-real violations case

In the PRV case, the eigenvectors  $x$  and eigenvalues  $\lambda$  of  $G(i\omega)$  satisfy

$$\begin{aligned} 2\lambda x &= [C(i\omega E - A)^{-1}B + D]x \\ &+ [B^T(-i\omega E^T - A^T)^{-1}C^T + D^T]x \end{aligned}$$

Putting

$$u = (i\omega E - A)^{-1} Bx, \quad v = (-i\omega E^T - A^T)^{-1} C^T x$$

we obtain

$$x = (2\lambda I_p - D - D^T)^{-1} [Cu + B^T v]$$

and hence

$$\begin{aligned} (i\omega E - A)u &= B(2\lambda I_p - D - D^T)^{-1} \\ &\times [Cu + B^T v] \\ (-i\omega E^T - A^T)v &= C^T(2\lambda I_p - D - D^T)^{-1} \\ &\times [Cu + B^T v] \end{aligned}$$

This means that the column vector  $(u, v)$  satisfies the generalized eigenvalue problem with eigenvalue  $i\omega$  :

$$i\omega \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathcal{B} & \mathcal{C} \\ \mathcal{D} & -\mathcal{B}^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad (1)$$

where the r.h.s. of (1) defines a Hamiltonian matrix with entries

$$\begin{aligned} \mathcal{B} &= A + B(2\lambda I_p - D - D^T)^{-1} C \\ \mathcal{C} &= B(2\lambda I_p - D - D^T)^{-1} B^T \\ \mathcal{D} &= -C^T(2\lambda I_p - D - D^T)^{-1} C \end{aligned}$$

Left-multiplying the l.h.s. and r.h.s. by the non-singular exchange matrix

$$\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

equation (1) can be written in the Hermitian symmetric matrix pencil  $(i\mathcal{A}, \mathcal{S})$  format [16] as :

$$(i\omega \mathcal{A} - \mathcal{S}) \begin{bmatrix} u \\ v \end{bmatrix} = 0 \quad (2)$$

where the skew-symmetric  $2n \times 2n$  matrix  $\mathcal{A}$  and the symmetric  $2n \times 2n$  matrix  $\mathcal{S}$  are given by

$$\mathcal{A} = \begin{bmatrix} 0 & E^T \\ -E & 0 \end{bmatrix}, \quad \mathcal{S} = \begin{bmatrix} \mathcal{D} & -\mathcal{B}^T \\ -\mathcal{B} & -\mathcal{C} \end{bmatrix}$$

The pertinent values for  $\omega$  are therefore obtained as the roots of the polynomial equation

$$\chi(\omega) = \det(i\omega\mathcal{A} - \mathcal{S})$$

which is a real and even polynomial in  $\omega$ , and hence the set of its roots is symmetric with respect to both the imaginary and real axes. Passivity violations occur when there are real generalized eigenvalues  $\omega$  pertaining to the eigenvalue  $\lambda = 0$ . These generalized eigenvalues allow accurate pinpointing of the cross-over frequencies which contain the exact boundaries of all passivity violation intervals. Note that the generalized eigenvalues can be judiciously calculated by means of the generalized Schur decomposition a.k.a. QZ decomposition [17] Sec. 7.7.

## 2.2. Bounded-real violations case

In the BRV case, we obtain similarly, adapting the approach of [6] to descriptor systems, the generalized eigenvalue problem

$$i\omega \begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \tilde{B} & \tilde{C} \\ \tilde{D} & -\tilde{B}^T \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \quad (3)$$

relating the angular frequency  $\omega$  and the singular values  $\sigma$ . The relevant Hamiltonian matrix entries are here

$$\tilde{B} = A - BR^{-1}D^TC$$

$$\tilde{C} = -\sigma BR^{-1}B^T$$

$$\tilde{D} = \sigma C^T S^{-1}C$$

where  $R = (D^TD - \sigma^2 I_p)$ ,  $S = (DD^T - \sigma^2 I_p)$  and the Hermitian matrix pencil equations (2) are formally the same with pencil  $(i\mathcal{A}, \tilde{\mathcal{S}})$ . Passivity violations occur when there are real generalized eigenvalues  $\omega$  pertaining to the singular value  $\sigma = 1$ .

## 3. SINGULAR DESCRIPTOR SYSTEMS

We use the term singular system when  $D + D^T$  is singular in the PRV case, and when  $D^TD - I_p$  is singular ( $DD^T - I_p$  is then singular as well), in the BRV case. In these cases the Hamiltonian

matrices of the previous section do not exist. However, these cases can be treated by the generalized eigenvalue approach [11] or the frequency inversion method [12].

### 3.1. Generalized eigenvalue method

In the PRV case, we adapt the argument of [11] (see also [18, 19] for connections with Riccati equations) for descriptor systems : it is easy to show that passivity is lost for values of  $\omega$  such that

$$\det[H(i\omega) + H^T(-i\omega)] = 0 \quad (4)$$

After some algebra, the zeros of determinant (4) can be obtained from

$$\det \begin{bmatrix} 0 & B & A - i\omega E \\ B^T & D + D^T & C \\ A^T + i\omega E^T & C^T & 0 \end{bmatrix} = 0$$

Hence it is clear that passivity violations occur as the purely imaginary eigenvalues of the generalized eigensystem

$$\begin{bmatrix} 0 & B & A \\ B^T & D + D^T & C \\ A^T & C^T & 0 \end{bmatrix} x = i\omega \begin{bmatrix} 0 & 0 & E \\ 0 & 0 & 0 \\ -E^T & 0 & 0 \end{bmatrix} x \quad (5)$$

Note that the right hand side matrix is singular in general, but this does not affect the solution of the generalized eigenvalue problem, since the algorithm we use is the QZ algorithm [20]. The singularity of the right hand side matrix yields infinite eigenvalues (corresponding to its null-space) which have to be discarded.

Similarly, in the BRV case, passivity is lost when

$$\det[I_p - H^T(-i\omega)H(i\omega)] = 0$$



After some easy but tiresome algebraic steps, we find that passivity violations occur as the purely imaginary eigenvalues of the generalized eigensystem

$$\begin{bmatrix} 0 & B & A & 0 \\ B^T & -I_p & 0 & D^T \\ A^T & 0 & 0 & C^T \\ 0 & D & C & -I_p \end{bmatrix} x = i\omega \begin{bmatrix} 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 \\ -E^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x$$

Performing some straightforward eliminations, we obtain the two equivalent more convenient generalized eigenvalue expressions :

$$\begin{bmatrix} 0 & B & A \\ B^T & D^T D - I_p & D^T C \\ A^T & C^T D & C^T C \end{bmatrix} x = i\omega \begin{bmatrix} 0 & 0 & E \\ 0 & 0 & 0 \\ -E^T & 0 & 0 \end{bmatrix} x \quad (6a)$$

$$\begin{bmatrix} BB^T & BD^T & A \\ DB^T & DD^T - I_p & C \\ A^T & C^T & 0 \end{bmatrix} x = i\omega \begin{bmatrix} 0 & 0 & E \\ 0 & 0 & 0 \\ -E^T & 0 & 0 \end{bmatrix} x \quad (6b)$$

### 3.2. Frequency inversion method

In [12] it was shown that, exploiting the fact that the 'reciprocal inverse' transfer function  $H(1/s)$  is positive-real resp. bounded-real whenever  $H(s)$  is positive-real resp. bounded-real, 'reciprocal inverse' matrices for testing passivity violations can be defined by means of the state space formulation for  $H(1/s) = C(s^{-1}I_n - A)^{-1}B + D$ , i.e.,

$$H(1/s) \Leftrightarrow (A^{-1}, -A^{-1}B, CA^{-1}, D - CA^{-1}B) \quad (7)$$

where  $(A, B, C, D)$  stands for the state space realization of  $H(s)$ . Of course, we have presupposed here that  $A$  is nonsingular, which is always the case when  $A$  is Hurwitz stable. Since the 'reciprocal inverse'  $D$ -matrix is  $D - CA^{-1}B$ , the Hamiltonian matrix pertaining to the 'reciprocal inverse' system is then in general better behaved than the Hamiltonian matrix pertaining to the original system. Of course, the passivity violations of the 'reciprocal inverse' system occur at the inverse

values  $1/\omega$  of those of the original system. This approach was tested successfully in [21].

The same can be done in the case of descriptor systems. It is not too hard to prove that the descriptor formulation of the system  $H(1/s) = C(s^{-1}E - A)^{-1}B + D$  is

$$H(1/s) \Leftrightarrow (E, -EA^{-1}B, C, D - CA^{-1}B, A) \quad (8)$$

where  $(A, B, C, D, E)$  stands for the descriptor state space realization of  $H(s)$ . As previously discussed, the Hamiltonian matrix pertaining to the 'reciprocal inverse' system is in general better behaved than the Hamiltonian matrix pertaining to the original system and the passivity violations of the 'reciprocal inverse' system occur at the inverse values  $1/\omega$  of those of the original system. But of course, the 'reciprocal inverse' system  $H(1/s)$  may itself be singular, in which case we must revert to the original generalized eigenvalue formulation.

#### 4. APPLICATION : SYMMETRIC SYSTEMS

Symmetric or reciprocal systems are of the most frequently encountered systems : this is in fact a consequence of the Lorentz reciprocity theorem [22]. In this section we will show how the descriptor state space methods of the two previous sections can be judiciously utilized in the case of symmetric state space systems.

Consider the state space system with minimal realization

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

and transfer function

$$H(s) = C(sI_n - A)^{-1}B + D$$

The system  $(A, B, C, D)$  is symmetric if  $H(s) = H(s)^T$ , i.e.,

$$D + C(sI_n - A)^{-1}B = D^T + B^T(sI_n - A^T)^{-1}C^T$$

for all  $s \in \mathbb{C} \setminus Sp(A)$ . We have the following :

*Theorem* : Let the minimal system  $(A, B, C, D)$  be symmetric. Then there exists an equivalent

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explicitly symmetric descriptor state space realization for the system, in other words matrices  $(A, B, C, D, E)$ , with  $A^T = A$ ,  $E^T = E$  nonsingular,  $D^T = D$  and  $C = B^T$ .

*Proof*

It is seen that for a symmetric system we must have  $\mathbf{D} = \mathbf{D}^T$ , hence we take  $D = \mathbf{D} = D^T$ . Also by [23], Theorem 6.2-4, or [24], Lemma 3, there exists a unique non-singular  $n \times n$  symmetric matrix  $P$  such that

$$\mathbf{A}^T = P^{-1}\mathbf{A}P, \quad \mathbf{C} = \mathbf{B}^T P^{-1}$$

Hence

$$H(s) = \mathbf{B}^T (sP - \mathbf{A}P)^{-1} \mathbf{B} + \mathbf{D}$$

If we take  $B = \mathbf{B}$ ,  $C = \mathbf{B}^T$ ,  $E = P = E^T$ ,  $A = \mathbf{A}P$ , and considering that  $A = \mathbf{A}P = PA^T = A^T$ , it is seen that the proof is complete.  $\square$

#### 4.1. Positive-real violations case

In the PRV case, equations (1) can be recast as

$$i\omega \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \frac{1}{2} \begin{bmatrix} K & K \\ -K & -K \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

where

$$K = K^T = B(\lambda I_p - D)^{-1}B^T$$

This can be decoupled to yield

$$i\omega E(u + v) = A(u - v) \quad (9)$$

$$i\omega E(u - v) = [A + K](u + v) \quad (10)$$

Since  $E$  is nonsingular, we can eliminate the  $u - v$  dependence from (9)-(10) yielding the simple eigenvalue equation

$$-\omega^2(u + v) = (E^{-1}AE^{-1})[A + K](u + v)$$

Since  $E^{-1}A = \mathbf{A}^T$  en  $E^{-1}B = \mathbf{C}^T$ , this can be further simplified in terms of the original system  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$  as

$$-\omega^2(u+v) = \mathbf{A}^T[\mathbf{A}^T + \mathbf{C}^T(\lambda I_p - \mathbf{D})^{-1}\mathbf{B}^T](u+v) \quad (11)$$

Passivity violations occur when there are real values  $\omega$  pertaining to the eigenvalue  $\lambda = 0$ , in other words if the matrix  $\mathbf{A}^T[\mathbf{A}^T - \mathbf{C}^T\mathbf{D}^{-1}\mathbf{B}^T]$  possesses real non-positive eigenvalues. Since the eigenvalues of a matrix and the eigenvalues of its transpose coincide, we have passivity violations if the matrix

$$T_1 = [\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}]\mathbf{A} \quad (12)$$

possesses real non-positive eigenvalues. Moreover, if we eliminate the  $u+v$  dependence from (9)-(10) and take the transpose, we find that there are passivity violations if the matrix

$$T_2 = \mathbf{A}[\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}] \quad (13)$$

possesses real non-positive eigenvalues. These results were also found by other means and in another context in [25, 26].

#### 4.2. Bounded-real violations case

In the BRV case, equations (3) can be recast as

$$i\omega \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} L & M \\ -M & -L \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

where

$$L = A - BR^{-1}DB^T$$

$$M = -\sigma BR^{-1}B^T$$

$$R = D^2 - \sigma^2 I_p$$

It should be noted that  $L, M, R$  are symmetric matrices. This can be decoupled to yield

$$i\omega E(u+v) = [L - M](u-v)$$

$$i\omega E(u-v) = [L + M](u+v)$$

Since  $E$  is nonsingular we can eliminate the  $u - v$  dependence and obtain the smaller  $n \times n$  symmetric generalized eigenvalue problem

$$-\omega^2 E(u + v) = [L - M]E^{-1}[L + M](u + v)$$

or

$$-\omega^2(u + v) = E^{-1}[L - M]E^{-1}[L + M](u + v)$$

This can further be simplified in terms of the original system  $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$  to

$$-\omega^2(u + v) = V(\sigma)V(-\sigma)(u + v)$$

where

$$V(\sigma) = \mathbf{A}^T - \mathbf{C}^T(\mathbf{D} + \sigma I_p)^{-1}\mathbf{B}^T \quad (14)$$

Passivity violations occur for  $\omega$  values corresponding to the singular value  $\sigma = 1$ , in other words if the matrix  $V(1)V(-1)$  possesses non-positive real eigenvalues. This is also true when the matrix  $V^T(-1)V^T(1)$ , i.e.,

$$(\mathbf{A} - \mathbf{B}(\mathbf{D} - I_p)^{-1}\mathbf{C})(\mathbf{A} - \mathbf{B}(\mathbf{D} + I_p)^{-1}\mathbf{C}) \quad (15)$$

possesses non-positive real eigenvalues. This result was also obtained by other means and in another context in [27].

#### 4.3. Strictly proper symmetric systems

It is seen from (11) in the PRV case, and (14) in the BRV case, that the passivity assessment is unfeasible in the case  $\mathbf{D} = 0$  (PRV strictly proper) and  $\mathbf{D} = \pm I_p$  (BRV strictly proper). By a slight abuse of language we will call all these systems strictly proper, since scattering matrices  $S(s)$  and immittance matrices  $H(s)$  are related through the functional equation<sup>†</sup>  $S(s) = \pm[I_p + H(s)]^{-1}[I_p - H(s)]$  [28].

<sup>†</sup>  $H(s)$  can be normalized by any positive quantity  $H_0$  of correct physical dimension, i.e.,  $H(s)$  may be replaced by  $H(s)/H_0$ .

4.3.1. *Positive-real violations case* In the PRV strictly proper case the transfer function is given by

$$H(s) = \mathbf{C}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B}$$

Passivity violations occur for the  $\omega$  values where  $H(i\omega) + H(-i\omega)$  is singular. It is easy to check whether  $\omega = 0$  violates passivity : this is the case when the symmetric matrix  $-\mathbf{CA}^{-1}\mathbf{B}$  is singular.

Next consider  $sH(s)$ , which can be written as

$$sH(s) = \mathbf{CB} + \mathbf{C}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{AB}$$

and in the same vein  $-s^2H(s)$ , i.e.,

$$-s^2H(s) = -s\mathbf{CB} - \mathbf{CAB} - \mathbf{CA}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{AB}$$

Since

$$\omega^2[H(i\omega) + H(-i\omega)] = H_1(i\omega) + H_1(-i\omega)$$

where

$$H_1(s) = -\mathbf{CAB} - \mathbf{CA}(s\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{AB} \quad (16)$$

non-zero passivity violations can be checked with respect to the new matrices  $(\mathbf{A}, \mathbf{AB}, -\mathbf{CA}, -\mathbf{CAB})$  by means of (12) and (13), provided of course that  $-\mathbf{CAB}$  is nonsingular, which we assume — see also [29, 30].

4.3.2. *Bounded-real violations case* Suppose first  $\mathbf{D} + I_p$  nonsingular. Putting  $\mathbf{A}_1 = \mathbf{A} - \mathbf{B}(\mathbf{D} + I_p)^{-1}\mathbf{C}$  we can rewrite matrix (15) as

$$(\mathbf{A}_1 - \mathbf{BD}_1^{-1}\mathbf{C})\mathbf{A}_1 \quad (17)$$

where  $\mathbf{D}_1 = 0.5(\mathbf{D}^2 - I_p)$ . This is easily derived from the identity

$$(\mathbf{D} - I_p)^{-1} - (\mathbf{D} + I_p)^{-1} = 2(\mathbf{D}^2 - I_p)^{-1}$$

Similarly, when  $\mathbf{D} - I_p$  is nonsingular, we put  $\mathbf{A}_2 = \mathbf{A} - \mathbf{B}(\mathbf{D} - I_p)^{-1}\mathbf{C}$  and we can now rewrite matrix (15) as

$$\mathbf{A}_2(\mathbf{A}_2 - \mathbf{BD}_2^{-1}\mathbf{C}) \quad (18)$$

where  $\mathbf{D}_2 = -0.5(\mathbf{D}^2 - I_p)$ . It is therefore clear that when  $\mathbf{D} = I_p$ , expression (17) is equivalent with expression (12) in the PRV—in the limit strictly proper— case. Similarly, when  $D = -I_p$ , expression (18) is equivalent with expression (13) in the PRV—in the limit strictly proper— case. As in subsection 4.3, in both cases we can eventually check the passivity violations by the PRV test with matrices  $(\mathbf{A}_k, \mathbf{A}_k\mathbf{B}, -\mathbf{C}\mathbf{A}_k, -\mathbf{C}\mathbf{A}_k\mathbf{B})$ , for  $k = 1, 2$ .

**4.3.3. General remark** It should be noted of course that the techniques discussed in section 3, i.e., the frequency inversion method and the generalized eigenvalue approach may also be used in the strictly proper symmetric case.

For example, in the PRV symmetric case, the generalized eigenvalue formulation becomes

$$\begin{bmatrix} 0 & B & A \\ B^T & 2D & B^T \\ A & B & 0 \end{bmatrix} x = i\omega \begin{bmatrix} 0 & 0 & E \\ 0 & 0 & 0 \\ -E & 0 & 0 \end{bmatrix} x$$

This can be shown to be equivalent with the smaller sized symmetric generalized eigenvalue problem :

$$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} x = -\omega^2 \begin{bmatrix} EA^{-1}E & 0 \\ 0 & 0 \end{bmatrix} x \quad (19)$$

or, reverting to the initial matrices, to the nonsymmetric generalized eigenvalue problem :

$$\begin{bmatrix} \mathbf{A}^2 & \mathbf{A}\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} x = -\omega^2 \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} x \quad (20)$$

Note that, when  $\mathbf{D}$  is nonsingular, this can be simplified to

$$T_2 x = -\omega^2 x$$

where  $T_2$  is the matrix of formula (13).

## 5. NUMERICAL EXAMPLES

For the sake of clarity, we start by summarizing the different cases and formulations in the following exhaustive table :

	1	2	3	4	5	6
	D/NS	D/S	FI	R/NS	R/S	R/SP
Y	(1-2)	(5)	(7-8)	(12-13)	(19-20)	(16)
S	(3-2)	(6)	(7-8)	(15)	(6)	(17-18)

The letters D and R stand for 'Descriptor system' and 'Reciprocal (symmetric) system', the letters Y and S stand for 'positive-real violations' and 'bounded-real violations', while NS, S, SP and FI respectively stand for 'Nonsingular', 'Singular', 'Strictly proper' and 'Frequency inversion'. For example, the label **Y1** is associated with the formulas (1-2), applicable to the PRV nonsingular descriptor case, while the label **S6** is associated with the formulas (17-18), applicable to the BRV symmetric strictly proper case. In the same vein, **Y3** and **S3** stand for the frequency inversion technique (7-8). All our examples treat the important singular cases 2,3,5,6, since the nonsingular cases have already been extensively treated in the literature [6, 25, 27].

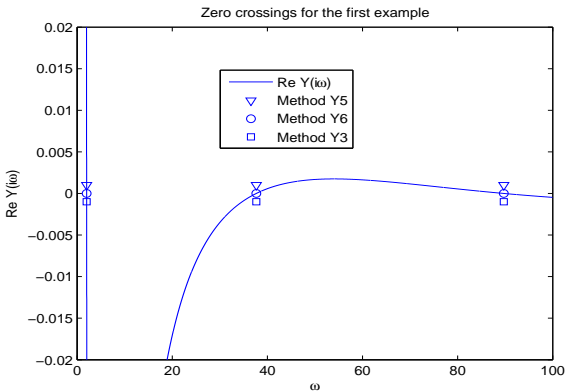


Figure 1. Zero crossings for the first example.

5.1. First example

As a first example we take the strictly proper SISO minimum-phase, but non-passive transfer function described in [31], i.e.,

$$Y(s) = \frac{(s + 25)(s + 35)(s + 38)(s + 180)(s + 185)}{(s + 1)(s + 3)(s + 90)^2(s + 95)(s + 100)}$$



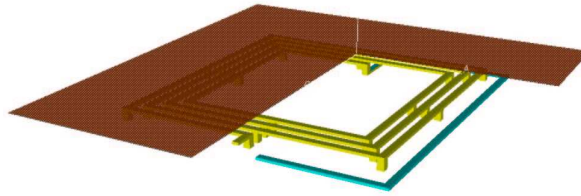


Figure 2. On-chip square spiral inductor.

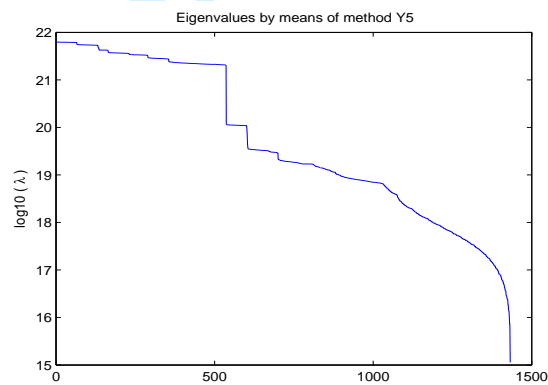


Figure 3. Y5 eigenvalues for the second example.

Utilizing the methods **Y3**, **Y5** and **Y6**, it is seen from Fig. 1 that the zero crossings are accurately pinpointed by all three methods.

### 5.2. Second example

As a second example we examine the potential passivity violations of the on-chip square spiral inductor described in [32]. The geometry of the spiral structure is shown in Fig. 2. The scalar admittance in descriptor state space format is given by

$$Y(s) = B^T(sE - A)^{-1}B$$

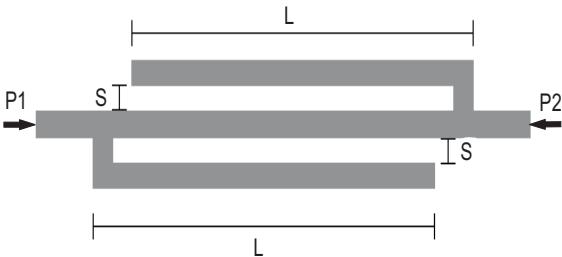


Figure 4. Double folded stub microstrip bandstop filter.

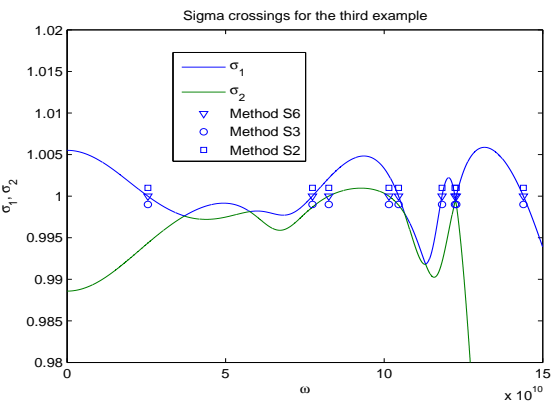


Figure 5. Sigma crossings for the third example.

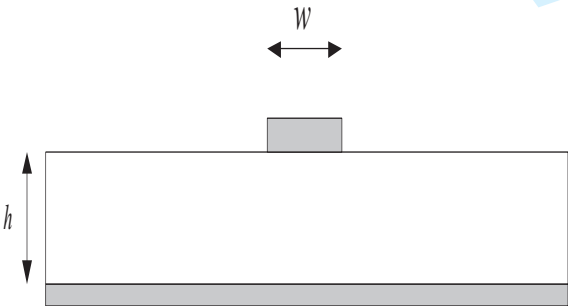


Figure 6. Microstrip geometry.

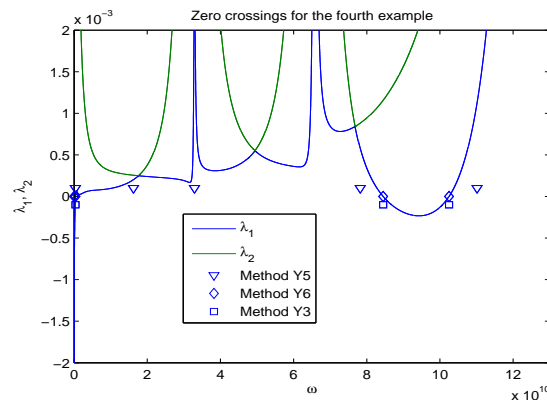


Figure 7. Zero crossings for the fourth example.

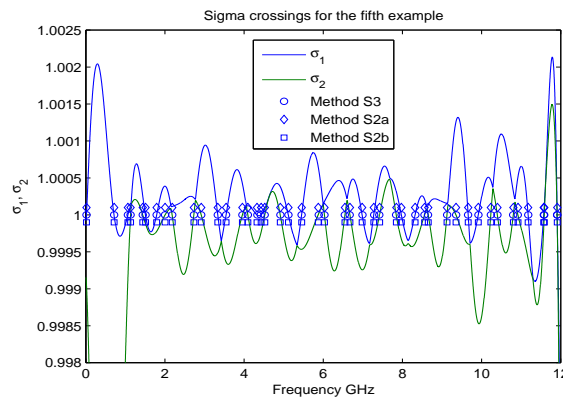


Figure 8. Sigma crossings for the fifth example. To be in accordance with the Figure of [21], the abscissa is marked in GHz frequency units.

where  $A$  and  $E$ , obtained by a filament PEEC method [32], are symmetric  $1434 \times 1434$  matrices with  $A$  sparse. In Fig. 3 we show the eigenvalues in logarithmic format obtained by method **Y5**, formula (19); since there are no negative eigenvalues we conclude immediately that the model for the spiral is passive.

### 5.3. Third example

In this example a double folded stub microstrip bandstop filter has been modelled [33]. The substrate is 0.1270 mm thick with a relative dielectric constant  $\epsilon_r = 9.9$  and a loss tangent  $\tan \delta = 0.003$ . The

length of each folded segment  $L$  is equal to  $1.97 \text{ mm}$ , while the varying spacing between a folded stub and the main line  $S$  is equal to  $0.117 \text{ mm}$ . The geometry is shown in Fig. 4. All data were simulated by ADS-Momentum<sup>‡</sup>. Utilizing the methods **S6**, **S3** and **S2**, it is seen from Fig. 5 that the sigma crossings<sup>§</sup> are accurately pinpointed by all three methods.

#### 5.4. Fourth example

In this example a microstrip line has been modelled by means of Vector Fitting, yielding a strictly proper admittance description. The conductor has width  $W = 100 \mu\text{m}$  and length  $L = 1.70 \text{ cm}$ ; the substrate has height  $h = 300 \mu\text{m}$ . The geometry is shown in Fig. 6. Utilizing the methods **Y3**, **Y5** and **Y6**, it is seen from Fig. 7 that the zero crossings are accurately pinpointed by methods **Y3** and **Y6**. Unfortunately, they are rather badly determined by method **Y5**, formula (20); we suspect that this may be partly due to the matrix squaring term  $\mathbf{A}^2$  present in (20).

#### 5.5. Fifth example

As a fifth and last example we examine the passivity violations of the quarter wavelength filter described in [21]. This example is highly pertinent since it is singular ( $D + I_p$  singular)—but not strictly proper—and since there are a lot of passivity violations. Utilizing the methods **S3**, **S2a** = (6a) and **S2b** = (6b), it is seen from Fig. 8 that the sigma crossings are accurately pinpointed by all three methods.

## 6. CONCLUSION

We have proposed general eigenvalue-based passivity tests for different practical cases of LTI state-space models. Passivity assessment methods have been developed for descriptor systems, including the important singular cases. We extended the already available techniques for standard state space systems in both positive-real and bounded-real violations cases. The theory was further applied to the important case of symmetric systems, since these systems can always be cast in an explicitly

<sup>‡</sup>Momentum EESof EDA, Agilent Technologies, Santa Rosa, CA.

<sup>§</sup>We call these sigma crossings, or crossings of 1, in order to distinguish with the zero crossings in the immittance case

reciprocal descriptor state space format. The strictly proper symmetric case has also been treated, since the classical Hamiltonian test matrices are undefined in that case. Finally, an exhaustive table of the different passivity assessment tests has been provided, and pertinent numerical examples have shown the applicability and accuracy of the different passivity assessment methods.

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